3.2 MATRIX Algebra (Suplement to Book slides)

#6) Given $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Determine if $B = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$ is a linear combination of A_1, A_2, A_3.

We want to find (if possible) C_1, C_2, and C_3 such that

$$C_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$$

which implies

$$\begin{bmatrix} C_1 + C_3 & -C_2 + C_3 \\ C_2 & C_1 + C_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -4 & 2 \end{bmatrix}$$

Equivalent to

$$\begin{cases} C_1 + 0C_2 + C_3 = 2 \\ 0C_1 - C_2 + C_3 = 3 \\ 0C_1 + C_2 + 0C_3 = -4 \\ C_1 + 0C_2 + C_3 = 2 \end{cases} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} \boxed{C_3 = -1} \\ \boxed{C_2 = -4} \\ C_1 = 2 - C_3 = \\ \text{or } \boxed{C_1 = 2 + 1 = 3} \end{array}$$

Are A_1, A_2 and A_3 lin. indep. ?

If

$$c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&-1\\1&0\end{bmatrix}+c_3\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

then

$$\begin{aligned}
c_1 + 0c_2 + c_3 &= 0\\
0c_1 - c_2 + c_3 &= 0\\
0c_1 + c_2 + 0c_3 &= 0\\
c_1 + 0c_2 + c_3 &= 0
\end{aligned}$$

$$\left[\begin{array}{ccc|c}1&0&1&0\\0&-1&1&0\\0&1&0&0\\1&0&1&0\end{array}\right]\sim\cdots\begin{bmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\Rightarrow \boxed{\begin{array}{l}c_3=0\\c_2=0\\c_1=0-c_3=0-0=0\end{array}}$$

Then. A_1, A_2 and A_3 are lin. indep.

Describe Span of A_1, A_2, A_3.

It consists of all matrices $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Then corresp augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 0 & w-z \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 0 & 2 & y+x \\ 0 & 0 & 1 & w-z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & -1 & 1 & x \\ 0 & 0 & 2 & y+x \\ 0 & 0 & 0 & y+x-2w+2z \end{array}\right]$$

Condition is

$$w - z = 0 \Rightarrow w = z.$$

$$(AB)^T = B^T A^T$$

<u>Example</u> $\quad A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 10 & 11 \\ 14 & 16 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} 10 & 14 \\ 11 & 16 \end{bmatrix}$$

Now

$$B^T A^T = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 11 & 16 \end{bmatrix}$$

Therefore, $\quad (AB)^T = B^T A^T.$

In general,

$$\left[(AB)^T\right]_{ij} = [AB]_{ji} = row_j(A) . col_i(B)$$

$$= col_j(A^T) \cdot row_i(B^T)$$

$$= row_i(B^T) \cdot col_j(A^T)$$

$$= \left[B^T A^T\right]_{ij}$$

Thm 3.4 (Properties of transpose)

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(kA)^T = k(A^T)$.

d) $(AB)^T = B^T A^T$

e) $(A^r)^T = (A^T)^r$.

Thm 3.5 $A_{n\times n}$

a) $A + A^T$ is symm.

b) AA^T and A^TA are symm.

Proof.

a) $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

b) $(AA^T)^T = (A^T)^T A^T = AA^T$.

In general,

if
$$A = [\vec{u}_1 \ \vec{u}_2 \ .. \ \vec{u}_n]$$

and
$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then,
$$A\vec{c} = c_1\vec{u}_1 + \cdot + c_n\vec{u}_n.$$

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Def a) $A.A = A^2$

b) $A^k = \overbrace{A \ldots A}^{k}$

Then if A square and $r, s > 0$ integers

$$A^r A^s = A^{r+s}$$

$$(A^r)^s = A^{rs}$$

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$A^3 = A^2A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

Then,

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$$

Def If $A_{m\times n}$, then

A^T is the matrix $n\times m$ obtained interchanging the rows and columns of A.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3}, \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}_{3\times 2}$$

Def. If $A_{n\times n}$, then A is symmetric if $A^T = A$.

Example.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 7 & 6 \\ 1 & 6 & 3 \end{bmatrix} = A^T.$$

Notice that
$$a_{12} = -1 = a_{21}$$
$$a_{13} = 1 = a_{31}$$
$$a_{23} = 6 = a_{32}$$

In general,
$$a_{ij} = a_{ji} \text{ in a symm. matrix.}$$